It would appear, then, that the "skin" and "core" could not produce the effect observed, and that if the two-material hypothesis is correct, then xylonite consists of an intimate mixture of two constituents, with different elastic and plastic properties and different stress-optical coefficients. It is further possible that the two materials are allotropic modifications of the same substance, and that their proportions are altered by the action of stress and other causes. This might account for the observed divergences in the values of α and β under different conditions.

The Rotation of Two Circular Cylinders in a Viscous Fluid.

By G. B. JEFFERY, M.A., D.Sc., Fellow of University College, London.

(Communicated by Prof. L. N. G. Filon, F.R.S. Received January 17, 1922.)

In a previous communication* we employed the solution of the equation $\nabla^4\psi = 0$ in bipolar co-ordinates defined by

$$\alpha + i\beta = \log \frac{x+i(y+a)}{x+i(y-a)} \tag{1}$$

to discuss the problem of the elastic equilibrium of a plate bounded by any two non-concentric circles.

There is a well-known analogy between plain elastic stress and two-dimensional steady motion of a viscous fluid, for which the stream-function satisfies $\nabla^4\psi = 0$. The boundary conditions are, however, different in the two cases, and the hydrodynamical problem has its own special difficulties.

In this paper we discuss the problem of the rotation of two parallel infinite circular cylinders in a viscous fluid. The solution is different in form according as one cylinder does, or does not, enclose the other. In the former case the problem can be solved in finite terms; while in the latter the problem is in general insoluble; that is to say, except in special circumstances, there is no *steady* motion which satisfies all the necessary conditions.

Let the two cylinders be defined by constant values of α, say, $\alpha = \alpha_1, \alpha_2$. We may take α_1 positive and greater than α_2, then α_2 will be positive or negative according as the second cylinder does, or does not, enclose the first. Let the cylinders rotate about their axes with angular velocities ω_1, ω_2. The stream-function ψ satisfies $\nabla^4\psi = 0$ when steady motion is established, and

* "Plane Stress and Plane Strain in Bipolar Co-ordinates," 'Phil. Trans.,' A, vol. 221, p. 265 (1920). Reference to this paper will be denoted by T.

will be an even function of β, single-valued for circuits which enclose either or both of the cylinders. Selecting the appropriate terms from our solution (T21), and re-instating terms explicitly rejected on the ground that they correspond to zero elastic stresses, we have*

$$
\begin{aligned}
h\psi = {} & A_0 \cosh\alpha + B_0\alpha\cosh\alpha + C_0\sinh\alpha + D_0\alpha\sinh\alpha \\
& + (A_1\cosh 2\alpha + B_1 + C_1\sinh 2\alpha + D_1\alpha)\cos\beta \\
& + \sum_{n=2}^{\infty}\{A_n\cosh(n+1)\alpha + B_n\cosh(n-1)\alpha + C_n\sinh(n+1)\alpha \\
& \qquad + D_n\sinh(n-1)\alpha\}\cos n\beta, \qquad (2)
\end{aligned}
$$

where $$h = (\cosh\alpha - \cos\beta)/a.$$

This solution is subject to the qualification (Tp. 275) that the series *may* converge only for positive or for negative values of α, or in other words, that the expansion of $h\psi$ may be different on opposite sides of the plane $\alpha = 0$.

Since ψ is single-valued it follows that the velocities are single-valued and that the stresses are single-valued if p the mean pressure is so. We can readily calculate p from (2) by noting that $\mu\nabla^2\psi$ and p are conjugate functions. In this way we find that p contains the many-valued term $(B_0 + D_1)\beta$, so that we must have

$$B_0 + D_1 = 0. \qquad (3)$$

§ 2. *A cylinder rotating in a viscous fluid contained in a non-concentric cylindrical vessel.* Let the inner cylinder be $\alpha = \alpha_1$ and the containing vessel $\alpha = \alpha_2$, and let their angular velocities ω_1, ω_2 be as shown in fig. 1. If d_1, d_2

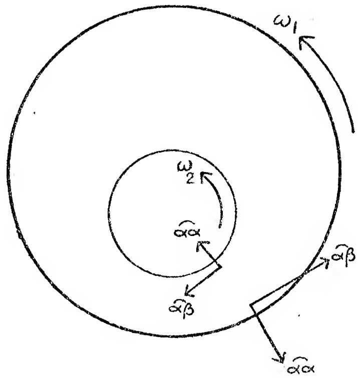

FIG. 1.

* Mr. B. Dutt ('Bull. Calcutta Math. Soc.,' vol. 10, p. 43 (1919) has given a solution of $\nabla^4\psi = 0$ in these co-ordinates and a discussion of the problem of the present paper. He, however, omits the terms in $a\cosh\alpha$, $a\sinh\alpha$, $a\cos\beta$, as well as some others which appear in our general solution (T 21). It will appear that these terms play an important part in the solution of our present problem. Moreover, Mr. Dutt takes n of the form half an odd integer, so that the series (2) has a discontinuity on the plane through the axes of the cylinders on part of which β has a discontinuity of 2π.

denote the distances of the centres of the circles α_1, α_2 from the origin, and r_1, r_2 their radii, we have (Tp. 284)

$$r_1 = a \operatorname{cosech} \alpha_1 \qquad r_2 = a \operatorname{cosech} \alpha_2,$$
$$d_1 = a \coth \alpha_1 \qquad d_2 = a \coth \alpha_2. \tag{4}$$

If the components of fluid velocity in the positive directions of α, β are u_α, u_β, we have

$$u_\alpha = -h\frac{\partial\psi}{\partial\beta} = -\frac{\partial}{\partial\beta}(h\psi) + \frac{\psi}{a}\sin\beta,$$
$$u_\beta = h\frac{\partial\psi}{\partial\alpha} = \frac{\partial}{\partial\alpha}(h\psi) - \frac{\psi}{a}\sinh\alpha, \tag{5}$$

and from these we obtain the boundary conditions

$$\left.\begin{aligned} \psi &= \psi_1 \\ \frac{\partial}{\partial\alpha}(h\psi) &= \frac{\psi_1}{a}\sinh\alpha_1 - \frac{a\omega_1}{\sinh\alpha_1} \end{aligned}\right\} \text{on } \alpha = \alpha_1 \tag{6}$$

and

$$\left.\begin{aligned} \psi &= \psi_2 \\ \frac{\partial}{\partial\alpha}(h\psi) &= \frac{\psi_2}{a}\sinh\alpha_2 - \frac{a\omega_2}{\sinh\alpha_2} \end{aligned}\right\} \text{on } \alpha = \alpha_2. \tag{7}$$

From (2) and (3) we assume

$$h\psi = A_0\cosh\alpha + B_0\alpha\cosh\alpha + C_0\sinh\alpha + D_0\alpha\sinh\alpha$$
$$+ (A_1\cosh 2\alpha + B_1 + C_1\sinh 2\alpha - B_0\alpha)\cos\beta, \tag{8}$$

and (5) and (6) then give

$$A_0\cosh\alpha_1 + B_0\alpha_1\cosh\alpha_1 + C_0\sinh\alpha_1 + D_0\alpha_1\sinh\alpha_1 = \cosh\alpha_1 \,.\, \psi_1/a,$$
$$A_0\sinh\alpha_1 + B_0(\alpha_1\sinh\alpha_1 + \cosh\alpha_1) + C_0\cosh\alpha_1$$
$$+ D_0(\alpha_1\cosh\alpha_1 + \sinh\alpha_1) = \frac{\psi_1}{a}\sinh\alpha_1 - \frac{a\omega_1}{\sinh\alpha_1},$$
$$A_1\cosh 2\alpha_1 + B_1 + C_1\sinh 2\alpha_1 - B_0\alpha_1 = -\psi_1/a,$$
$$2A_1\sinh 2\alpha_1 + 2C_1\cosh 2\alpha_1 - B_0 = 0, \tag{9}$$

with four precisely similar equations obtained from these by writing $\alpha_2, \psi_2, \omega_2$ for $\alpha_1, \psi_1, \omega_1$ respectively. We have thus eight equations for the seven coefficients and the condition for their consistence will determine $\psi_1 - \psi_2$.

With the usual notation for the stresses,

$$\widehat{\alpha\beta} = \mu\left\{\frac{\partial}{\partial\alpha}\left(h^2\frac{\partial\psi}{\partial\alpha}\right) - \frac{\partial}{\partial}\left(h^2\frac{\partial\psi}{\partial\beta}\right)\right\}, \tag{10}$$

where μ is the coefficient of viscosity. On a surface for which α, ψ are constant this reduces to

$$\widehat{\alpha\beta} = \mu h\left\{\frac{\partial^2}{\partial\alpha^2}(h\psi) - \frac{\psi}{a}\cosh\alpha\right\}. \tag{11}$$

The couple per unit length which must be applied to the inner cylinder in order to maintain the motion is

$$L_1 = -a \operatorname{cosech} \alpha_1 \int_{-\pi}^{+\pi} \frac{\widehat{\alpha\beta}}{h} \, \alpha\beta$$

and using (11), (8) and the first of equations (9), this reduces to

$$L_1 = -4\pi\mu a\{B_0 + D_0 \coth \alpha_1\}. \tag{12}$$

Similarly the couple which must be applied to the containing vessel is

$$L_2 = 4\pi\mu a\{B_0 + D_0 \coth \alpha_2\}. \tag{13}$$

The couple is in each case reckoned positive in the same sense as the angular velocity.

Solving equations (9) and substituting in (12) and (13), we obtain

$$L_1 = \frac{4\pi\mu a^2\{(\omega_1-\omega_2)(\alpha_1-\alpha_2)\coth(\alpha_1-\alpha_2) - \omega_1 \sinh \alpha_2 \operatorname{cosech}^2 \alpha_1 \sinh(2\alpha_1-\alpha_2) + \omega^2\}}{(\alpha_1-\alpha_2)\{\sinh^2\alpha_1 + \sinh^2\alpha_2\} - 2\sinh\alpha_1 \sinh\alpha_2 \sinh(\alpha_1-\alpha_2)} \tag{14}$$

and

$$L_2 = -\frac{4\pi\mu a^2\{(\omega_1-\omega_2)(\alpha_1-\alpha_2)\coth(\alpha_1-\alpha_2) - \omega_1 - \omega_2 \sinh \alpha_1 \operatorname{cosech}^2 \alpha_2 \sinh(\alpha_1-2\alpha_2)\}}{(\alpha_1-\alpha_2)\{\sinh^2\alpha_1 + \sinh^2\alpha_2\} - 2\sinh\alpha_1 \sinh\alpha_2 \sinh(\alpha_1-\alpha_2)}. \tag{15}$$

These rather complicated formulæ take a very simple form in a special case. If we put $\omega_2 = 0$, $\alpha_2 = 0$, we have the solution for a cylinder rotating in a viscous fluid bounded by an infinite rigid plane parallel to the axis of the cylinder. The couple required to keep the cylinder in motion is obtained from (14). Using (4), we may express this in terms of the radius of the cylinder (r), and the distance of its axis from the plane boundary (d). We obtain

$$L_1 = 4\pi\mu\omega_1 r^2 d/\sqrt{(d^2-r^2)}. \tag{16}$$

Making d tend to infinity, we have the well-known value of the couple necessary to maintain the rotation of a cylinder in an infinite fluid, namely, $4\pi\mu\omega_1 r^2$, per unit length.

§ 3. *The rotation of two cylinders in an infinite viscous fluid.* In this case there is, in general, no steady motion of the fluid for which the velocity of the fluid vanishes at infinity. This may be shown by means of (2) for any radii and angular velocities of the cylinders, if we allow for the difference in the expansions on opposite sides of the plane $\alpha = 0$. But the algebra is heavy and we will confine ourselves to the consideration of a particular case which brings out quite clearly the nature of the difficulty.

Let two equal circular cylinders $\alpha = \pm\alpha_1$ rotate in opposite senses with

angular velocity ω as shown in fig. 2. We may then avoid all difficulties arising from the possibility of different expansions for $h\psi$ according as $\alpha \gtrless 0$

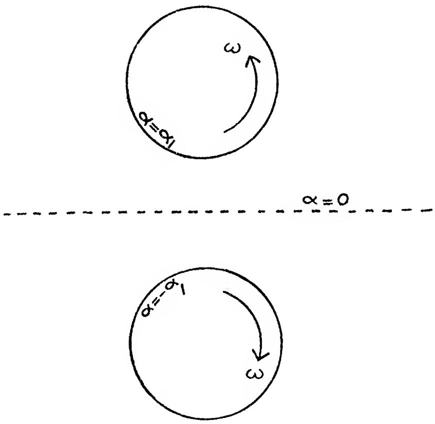

FIG. 2.

by noting that, by symmetry, ψ is a constant and $\widehat{\alpha\beta}$ vanishes on the plane $\alpha = 0$. Using (11), we may take

$$\psi = 0, \qquad \frac{\partial^2 (h\psi)}{\partial \alpha^2} = 0 \tag{17}$$

on $\alpha = 0$. It may be seen that the most general form of (2) consistent with (17) is

$$h\psi = B_0 \alpha (\cosh \alpha - \cos \beta) + C_0 \sinh \alpha + C_1 \sinh 2\alpha \cos \beta. \tag{18}$$

The boundary conditions (6) give four equations to determine B_0, C_0, C_1, ψ_1. They are consistent and they give

$$B_0 = -\frac{a\omega \cosh 2\alpha_1}{2 \cosh \alpha_1 \sinh^3 \alpha_1}, \quad C_0 = \frac{a\omega \cosh \alpha_1}{2 \sinh^3 \alpha_1}, \quad C_1 = -\frac{a\omega}{4 \cosh \alpha_1 \sinh^3 \alpha_1}.$$

At infinity (α, $\beta = 0$), we have from (5)

$$u_\alpha = 0$$

$$u_\beta = C_0 + 2C_1 = \frac{a\omega}{2 \cosh \alpha_1 \sinh \alpha_1},$$

so that, if r is the radius of either cylinder and d is the distance apart of their axes, we see using (4) that the fluid at a distance from the cylinders is moving in uniform stream motion with velocity V in a direction perpendicular to the plane containing the axes, where

$$V = r^2 \omega / d. \tag{19}$$

The above demonstration is open to the criticism that in obtaining (18) we have assumed that (2) is absolutely and uniformly convergent *on* the plane

$\alpha = 0$, which may possibly be the boundary of its region of convergence. This difficulty may be avoided and some further light thrown upon the physical nature of the problem by proceeding as follows: we can uniquely determine ψ in the form (8) so that it satisfies the boundary conditions (6) on $\alpha = \alpha_1$ and (17) on $\alpha = \alpha_2$, where α_2 is positive. This would correspond to the stream-function due to a cylinder rotating in a viscous fluid contained in a non-concentric cylindrical vessel whose inner surface is smooth and incapable of exerting any tangential traction on the fluid. This problem admits of a unique solution. Now make α_2 tend to zero so that the radius of the containing vessel becomes very large. It will be found that the motion of the fluid in those parts of the vessel which are remote from the rotating cylinder does not tend to zero, but in fact tends to the motion described by (19).

The kinetic energy of the motion is clearly infinite while the couple on the cylinder is easily seen to be finite. It follows that the motion could not be set up in any finite time. No steady motion will ever be reached, and the longer the cylinders have been rotating the greater the mass of fluid which is set in motion.

This result is not surprising. By a well-known result due to Stokes, the motion of a viscous fluid due to a single circular cylinder moving through it with a velocity of translation never attains to a steady state. And our present problem is very similar to that of Stokes. Both are concerned with "doublet" motions, in which fluid is forced out in one direction and drawn in in the opposite direction.